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Lyapunov's Theorem for Measures on D-posets[1]

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We generalize Lyapunov's convexity theorem for measures on effect algebras.

KEY WORDS: Lyapunov theorem; effect algebra; measure.

1. INTRODUCTION

One of the most loved and celebrated theorems of measure theory is "Lyapunov's theorem" which states that the range of a nonatomic σ-additive measure on a σ-algebra with values in a finite dimensional vector space is convex. In this paper I extend Lyapunov's theorem to measures defined on D-posets. My proof follows Halmos' idea and reduces the proof to the semi-convex case. Indeed, I extend Lyapunov's theorem to measures defined on a weaker algebraic structure which is endowed with an ordering and a partial operation compatible with order as explained in Section 4. Every D-poset satisfies these axioms and so does every complemented modular lattice. So Theorem 4.10 generalizes Avallone's version, (Avallone, 1995) Theorem 2.3, of Lyapunov's theorem valid for modular function on complemented lattices.

In (Avallone and Basile, 2003), Avallone and Basile have applied the version of Lyapunov's theorem valid for D-posets, Theorem 3.6 in an economic context.

D-posets have been introduced in Chovanec and Kopka (1994) as a generalization of many structures, as orthomodular posets, orthoalgebras and MV-algebras. Therefore the study of measures on D-posets allows us to unify the study of measures on orthomodular posets in noncommutative measure theory and measures on MV-algebras in fuzzy measure theory.

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2. PRELIMINARIES

We will fix some notations. First of all, we will give the definition of a D-poset. Examples of D-posets can be found in Chovanec and Kopka (1994).

Definition 2.1. Let $(P, \leq)$ be a partial ordered set (for short poset). A partial binary operation $\ominus$ on P such that $b \ominus a$ is defined if and only if $a \leq b$ is called a difference on $(P, \leq)$ if the following conditions are satisfied for all $a,\ b,\ c \in P$:

(1) If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$
(2) If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Definition 2.2. Let $(P, \leq, \ominus)$ be a poset with difference and let 0 and 1 be the smallest and greatest elements in P, respectively. The structure $(P, \leq, \ominus)$ is called a *difference poset (D-poset* for short), or a *difference lattice (D-lattice* for short) if P is a lattice.

An alternative structure to a D-poset is that of an effect algebra introduced by Foulis and Bennett in (Foulis and Bennett, 1994). These two structures, D-posets and effect algebras, are equivalent as shown in (Dvurečenskij and Pulmannová, 2000), Theorem 1.3.4.

From now on, P denotes a D-poset.

If $a \in P$, we set $a^\perp = 1 \ominus a$.

We say that a and b are orthogonal if $a \leq b^\perp$ and we write $a \perp b$. If $a \perp b$, we set $a \oplus b = (a^\perp \ominus b)^\perp$. If $a_1, \ldots, a_n \in P$ we inductively define $a_1 \oplus \ldots \oplus a_n = (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$ if the right side exists. The sum is independent on any permutation of the elements. We say that $\{a_1, \ldots, a_n\}$ is orthogonal if $a_1 \oplus \ldots \oplus a_n$ exists. If $a \in P$, a partition of a is an orthogonal family $\{a_1, \ldots, a_n\}$ with $\oplus_{i \leq n} a_i = a$.

Proposition 2.1.

(1) *If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.*
(2) *If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$*
(3) *If $a \leq b$, then $b = a \oplus (b \ominus a)$.*

In the following, we use a property weaker than Dedekind σ-completeness:

P has the *interpolation property* if, for all sequences x_n, y_n in P with $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$, $(n \in \mathbb{N})$ there exists $x \in P$ such that $x_n \leq x \leq y_n$ for every $n \in \mathbb{N}$.

A function μ on P with values in a linear space E is called *measure* if $a \perp b$ implies $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure iff $a \leq b$ implies $\mu(b \ominus a) = \mu(b) - \mu(a)$. We say that μ is *semi-convex* if, for every $a \in P$, there exists $b \leq a$ such that $\mu(b) = \frac{1}{2}\mu(a)$.

A function ν on a lattice is called *modular* if $\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y)$.
We now give two properties for measures on D-posets.

According to Weber's terminology (Weber, 1996) we extend the definition of μ-chained.

Definition 2.3. Let μ be a function on a poset L with values in a linear normed space. We say that L is μ-chained if for every $\varepsilon > 0$ and for every $a, b \in L$ with $a \le b$ there exist $x_0, x_1, \ldots, x_n \in L$ such that $a = x_0 \le x_1 \le \ldots \le x_n = b$ and $|\mu(c) - \mu(d)| < \varepsilon$ whenever $c, d \in [x_i, x_{i+1}]$ for $i = 0, \ldots, n-1$.

According to Bhaskara Rao and Bhaskara Rao's terminology (Bhaskara Rao and Bhaskara Rao, 1983) we extend the definition of μ strongly continuous.

Definition 2.4. Let μ be a function on P with values in a linear normed space. We say that μ is strongly continuous if for every $\varepsilon > 0$ and $a \in L$, there exists a partition $\{a_1, \ldots, a_m\}$ of a in L such that $|\mu(b)| < \varepsilon$ whenever $b \le a_j$, $j \le m$.

We compare these two properties.

Proposition 2.2. *Let $\mu : P \to E$ be a measure on P with values in a linear normed space. Then μ is strongly continuous iff P is μ-chained.*

Proof: $\Rightarrow$ Let $\varepsilon > 0$ and $a, b \in P$ with $a \le b$.

By hypothesis there exist $a_1, \ldots, a_m \in P$ such that $a_1 \oplus \ldots \oplus a_m = b \ominus a$ and $|\mu(e)| < \frac{\varepsilon}{2}$ whenever $e \le a_j$, $j \le m$. Put $x_i := a \oplus a_1 \oplus \ldots \oplus a_i$ for $i = 1, \ldots, m$ and $x_0 := a$. Then $a = x_0 \le x_1 \le \ldots \le x_m = b$. Let $c, d \in [x_i, x_{i+1}]$. Then $c \ominus x_i \le x_{i+1} \ominus x_i = a_{i+1}$ and $d \ominus x_i \le x_{i+1} \ominus x_i = a_{i+1}$. So $|\mu(d \ominus x_i)| < \frac{\varepsilon}{2}$ and $|\mu(c \ominus x_i)| < \frac{\varepsilon}{2}$. Thus $|\mu(c) - \mu(d)| = |\mu(c \ominus x_i) - \mu(d \ominus x_i)| < \varepsilon$.

$\Leftarrow$ Let $\varepsilon > 0$ and $a \in P$. By hypothesis there exist $x_0, x_1, \ldots, x_n \in P$ such that $0 = x_0 \le x_1 \le \ldots \le x_n = a$ and $|\mu(c) - \mu(d)| < \varepsilon$ whenever $c, d \in [x_i, x_{i+1}]$. Put $a_i := x_i \ominus x_{i-1}$ for $i = 1, \ldots, n$. Then $\{a_1, \ldots, a_n\}$ is an orthogonal family and $a_1 \oplus \ldots \oplus a_n = a$. Moreover, if $b \le a_i$, then $x_{i-1} \le b \oplus x_{i-1} \le x_i$. Hence $|\mu(b)| = |\mu(b \oplus x_{i-1}) - \mu(x_{i-1})| < \varepsilon$. $\square$

3. THE MAIN RESULT

The proof of Lyapunov's theorem 3.8, the main result of this paper, is based on a series of lemmata. Now we study conditions which ensures convexity of the range.

Lemma 3.1. *Let L be a poset with 0 which satisfies the interpolation property and $\nu : L \to \mathbb{R}$ be a monotone function. Suppose that L is ν-chained, then $\nu(L)$ is an interval.*

Proof: Let $a \in L$, pick an increasing sequence $(C_n)_{n\in\mathbb{N}}$ of finite chains from 0 to a with the following property:

$|\nu(x) - \nu(y)| < \frac{1}{n}$ whenever x and y are two consecutive elements of C_n.

Put C a maximal chain with $\cup_{n\in\mathbb{N}} C_n \subset C \subset [0, a]$. Since $\nu(\cup_{n\in\mathbb{N}} C_n)$ is dense in $[\nu(0), \nu(a)]$, so is $\nu(C)$. We will prove $\nu(C) = [\nu(0), \nu(a)]$. Let $r \in [\nu(0), \nu(a)]$, $D_1 := \{x \in C : \nu(x) \leq r\}$ and $D_2 := \{x \in C : \nu(x) \geq r\}$. Since $\nu(C)$ is dense in $[\nu(0), \nu(a)]$, $r = \sup_{x\in D_1} \nu(x) = \inf_{x\in D_2} \nu(x)$. Choose an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in D_1 such that $\nu(x_n) \to r$ and a decreasing sequence $(y_n)_{n\in\mathbb{N}}$ in D_2 such that $\nu(y_n) \to r$. By the maximality of C, also C has the interpolation property. Hence there exists $x \in C$ such that $x_n \leq x \leq y_n$. Therefore we get $\nu(x) = r$. □

Lemma 3.2. *Let L be a poset with the interpolation property, E be a Archimedean Riesz space, $\nu : L \to E$ be a monotone function such that for all $a, b \in L$, $a \leq b$, there exists $c \in [a, b]$ with $\nu(c) = \frac{\nu(a)+\nu(b)}{2}$. Then for all $a, b \in L$, $a \leq b$, there exists a monotone function defined on the real unit interval $\gamma : [0, 1] \to [a, b]$ such that $\gamma(0) = a$, $\gamma(1) = b$ and*

$$\nu(\gamma(t)) = t\nu(b) + (1 - t)\nu(a). \qquad (*)$$

Proof: We inductively define γ for every rational dyadic number $r \in [0, 1]$.

For $n \in \mathbb{N} \cup \{0\}$ put $T_n := \{\frac{i}{2^n} \mid \; i = 0, \ldots, 2^n\}$.

On $T_0 = \{0, 1\}$ we define γ by $\gamma(0) := a$ and $\gamma(1) := b$.

Suppose that we have defined γ on T_n as required. Let $r \in T_{n+1} \setminus T_n$, then $r = \frac{2k+1}{2^{n+1}}$ for $k = 0, \ldots, 2^n - 1$. We observe that $\frac{2k+1}{2^{n+1}} = \frac{k}{2^n} + \frac{1}{2^{n+1}}$. By hypothesis there exists $c \in [\gamma(\frac{k}{2^n}), \gamma(\frac{k+1}{2^n})]$ with $\nu(c) = \frac{\nu(\gamma(\frac{k}{2^n}))+\nu(\gamma(\frac{k+1}{2^n}))}{2}$. Then we choose $\gamma(r) = c$.

Now we have to define $\gamma(t)$ for $t \in [0, 1]$.

Put $r_n := \min\{r \in T_n : r \geq t\}$ and $s_n := \max\{r \in T_n : r \leq t\}$. Then $(r_n)_{n\in\mathbb{N}}$ is a decreasing sequence of rational dyadic number with infimum t and $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence of rational dyadic number with supremum t. Since L has the interpolation property, there exists $z \in L$ with $\gamma(r_n) \leq z \leq \gamma(s_n)$ and we choose $\gamma(t) := z$. Now we check that z is as required. By monotonicity we have $r_n\nu(b) + (1 - r_n)\nu(a) = \nu(\gamma(r_n)) \leq \nu(\gamma(t)) \leq \nu(\gamma(s_n)) = s_n\nu(b) + (1 - s_n)\nu(a)$. Passing to the order limit for n which tends to infinity and observing that E is Archimedean we have formula (*). □

Lemma 3.3. *Let P be a D-poset with the interpolation property, E be an Archimedean Riesz space and μ a positive measure on P with values in E. If μ is semi-convex, then for all $a, b \in P$ with $a \leq b$ there exists a monotone map defined on the real unit interval $\gamma : [0, 1] \to [a, b]$ such that $\gamma(0) = a$, $\gamma(1) = b$ and $\mu(\gamma(t)) = (1 - t)\mu(a) + t\mu(b)$ for every $t \in [0, 1]$.*

Proof: First observe that μ is monotone.

We will prove that the assumptions of Lemma 3.2 are fulfilled.

Let $a, b \in P$, with $a \leq b$, consider c such that $a \oplus c = b$ and choose $c' \leq c$ such that $\mu(c') = \frac{1}{2}\mu(c)$. Then for $d = a \oplus c'$, we have $\mu(d) = \mu(a \oplus c') = \mu(a) + \frac{1}{2}\mu(b) - \frac{1}{2}\mu(a) = \frac{1}{2}(\mu(a) + \mu(b))$. Now the proof follows from Lemma 3.2. □

Lemma 3.4. *Let E be a linear space and $\mu : P \to E$ a measure. Let a_1, a_2 be two orthogonal elements of P. Let $\gamma_i : [0, 1] \to [0, a_i]$ be functions with $\gamma_i(t) = t\mu(a_i)$ for $t \in [0, 1]$, $i = 1, 2$. Then $\gamma(t) = \gamma_1(1-t) \oplus \gamma_2(t)$ is well-defined and $\mu(\gamma(t)) = t\mu(a_2) + (1-t)\mu(a_1)$.*

The following fundamental trick, also used in Armstrong and Prikry (1981), Candeloro and Sacchetti (1979), Volkmer and Weber (1983), comes from Halmos.

Lemma 3.5. *Let P be a D-poset with the interpolation property, E be an Archimedean Riesz space and μ a positive measure on P with values in E. Let $\nu : P \to [0, +\infty)$ be a measure such that $\nu(t_n) \to 0$ whenever $\mu(t_n)$ is order convergent to 0. If μ is semi-convex, then (μ, ν) is semi-convex.*

Proof: Let $a \in P$ and $\mu' := (\mu, \nu)$. Since μ is semi-convex there exists $a_1 \leq a$ such that $\mu(a_1) = \frac{1}{2}\mu(a)$. Let a_2 such that $a_1 \oplus a_2 = a$. By Lemma 3.3 there exists a monotone function $\gamma_1 : [0, 1] \to P$ such that $\gamma_1(0) = 0$, $\gamma_1(1) = a_1$ and $\mu(\gamma_1(t)) = t\mu(a_1)$.

We prove that $\nu \circ \gamma_1$ is continuous:

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in [0,1] converging to t. Suppose first that $t_n \leq t$ for every $n \in \mathbb{N}$. Since $\gamma_1(t_n) \leq \gamma_1(t)$, then there are $c_n \in P$ such that $\gamma_1(t_n) \oplus c_n = \gamma_1(t)$. Since $\mu(c_n) = \mu(\gamma_1(t)) - \mu(\gamma_1(t_n)) = (t - t_n)\mu(a_1) \to 0$, we obtain $\nu(\gamma_1(t)) - \nu(\gamma_1(t_n)) = \nu(c_n) \to 0$. Analogously, one treats the case $t_n \geq t$ for $n \in \mathbb{N}$.

Again by Lemma 3.3 there exists a monotone function $\gamma_2 : [0, 1] \to P$ such that $\gamma_2(0) = 0$, $\gamma_2(1) = a_2$ and $\mu(\gamma_2(t)) = t\mu(a_2)$. As we have seen before $\nu \circ \gamma_2$ is continuous. Choose γ as in Lemma 3.4. Then $\nu \circ \gamma(t) = \nu \circ \gamma_1(1-t) + \nu \circ \gamma_2(t)$ is continuous.

Since ν is additive on orthogonal elements, $\nu(a_1) \leq \frac{1}{2}\nu(a) \leq \nu(a_2)$ or $\nu(a_2) \leq \frac{1}{2}\nu(a) \leq \nu(a_1)$. By the continuity of $\nu \circ \gamma$ there exists $t_0 \in [0, 1]$ such that $\nu(\gamma(t_0)) = \frac{1}{2}\nu(a)$.

Since $\mu(a_1) = \mu(a_2) = \frac{1}{2}\mu(a)$, we have $\mu(\gamma(t)) = (1-t)\mu(a_1) + t\mu(a_2) = \frac{1}{2}\mu(a)$.

Put $d := \gamma(t_0)$, then we have $d \leq a$ and $\mu'(d) = \frac{1}{2}\mu'(a)$. □

The following theorem has been used by Avallone and Basile in (Avallone and Basile, in press).

Theorem 3.6. *Let P be a D-poset with the interpolation property, $\mu : P \to \mathbb{R}^l$ a positive strongly continuous measure. Then for all $a,\ b \in P$ with $a \perp b$ and for all $a,\ b \in P$ with $a \leq b$ the segment $\overline{\mu(a), \mu(b)}$ is contained in the range of μ; in particular the range of μ is a star-shaped domain with respect to* 0.

Proof: First, we will show that μ is semi-convex by induction on l.
For $l = 1$ the proof follows by Lemma 3.1 and Proposition 2.2. Assume that $l \geq 2$ and the theorem is true for $\mathbb{R}^{l-1}$-valued measures.
Put $\mu' := (\mu_1 + \mu_l, \mu_2, \ldots, \mu_{l-1})$. By assumptions μ' is semi-convex. Since $\mu'(x_n) \to 0$ implies $\mu_l(x_n) \to 0$, we have by Lemma 3.5 that (μ', μ_l) is semi-convex. Put $\nu := (\mu', \mu_l)$. Let $T : (z_1, \ldots, z_l) \in \mathbb{R}^l \mapsto (z_1 - z_l, z_2, \ldots, z_l) \in \mathbb{R}^l$. Then T is a linear map and $T \circ \nu = \mu$. It follows that μ is semi-convex.

Now the rest of the proof comes from Lemma 3.3 and 3.4. $\square$

Not all strongly continuous measures on orthomodular lattices have convex range:

Example 3.5. Let L be an orthomodular lattice, $L = B_1 \cup B_2$, where $B_1 \cap B_2 = \{0, 1\}$ and B_i are two complete blocks on which are defined atomless positive σ-additive measures μ_i with values in $\mathbb{R}^2$ and $\mu_1(1) = \mu_2(1)$. Let $\mu : L \to \mathbb{R}^2$ be the measure defined by $\mu|_{B_i} := \mu_i$. Then the range of μ is the union of two convex sets, so it is in general not convex:

For instance, take atomless positive real-valued measures λ_i on B_i with $\lambda_i(1) = 2$ and vectors $e_i,\ f_i \in \mathbb{R}^2$ with $e_1 + e_2 = f_1 + f_2$. Moreover, let $a_i \in B_i$ with $\lambda_i(a_i) = 1$ and $a_i^{\perp}$ be the complement of a_i in B_i. Define $\mu_i : B_i \to \mathbb{R}^2$ by $\mu_i(x) = \lambda_i(x \wedge a_i)e_i + \lambda_i(x \wedge a_i^{\perp})f_i$. Then $\mu(L)$ is the union of the parallelograms $\mu_1(B_1)$ and $\mu_2(B_2)$ generated by $e_1,\ f_1$ and $e_2,\ f_2$, respectively. Hence $\mu(L)$ is not convex, e.g. if $e_1 = (0, 1),\ f_1 = (2, 1),\ e_2 = (1, 0),\ f_2 = (1, 2)$.

But modularity forces convexity! We need some preparatory stuff.

Lemma 3.7. *Let L be a lattice, E a linear space and $\mu : L \to E$ a modular function. Suppose that for all $a,\ b \in L$ with $a \leq b$ there exists $\gamma : [0, 1] \to [a, b]$ such that $\gamma(0) = a,\ \gamma(1) = b$ and $\mu(\gamma(t)) = t\mu(b) + (1 - t)\mu(a)$. Then for all $a,\ b \in L$ there exists $\gamma : [0, 1] \to [a \wedge b, a \vee b]$ such that $\gamma(0) = a,\ \gamma(1) = b$ and $\mu(\gamma(t)) = (1 - t)\mu(a) + t\mu(b)$.*

Proof: Let $\gamma_1 : [0, 1] \to [a \wedge b, a]$ such that $\gamma_1(0) = a \wedge b,\ \gamma_1(1) = a$ and $\mu(\gamma_1(t)) = (1 - t)\mu(a \wedge b) + t\mu(a)$. Let $\gamma_2 : [0, 1] \to [a \wedge b, b]$ such that $\gamma_2(0) = a \wedge b,\ \gamma_2(1) = b$ and $\mu(\gamma_2(t)) = (1 - t)\mu(a \wedge b) + t\mu(b)$.

Put $\gamma(t) := \gamma_1(1-t) \vee \gamma_2(t) \in [a \wedge b, a \vee b]$. Then $\gamma(0) = \gamma_1(1) \vee \gamma_2(0) = a \vee (a \wedge b) = a$, $\gamma(1) = \gamma_1(0) \vee \gamma_2(1) = b$ and moreover by modularity $\mu(\gamma(t)) = \mu(\gamma_1(1-t)) + \mu(\gamma_2(t)) - \mu(\gamma_1(1-t) \wedge \gamma_2(t))$. We observe $\gamma_1(1-t) \in [a \wedge b, a]$ and $\gamma_2(t) \in [a \wedge b, b]$, so $\gamma_1(1-t) \wedge \gamma_2(t) = a \wedge b$. Therefore $\mu(\gamma(t)) = t\mu(a \wedge b) + (1-t)\mu(a) + (1-t)\mu(a \wedge b) + t\mu(b) - \mu(a \wedge b) = (1-t)\mu(a) + t\mu(b)$. □

Definition 3.6. Let $\mu : P \to \mathbb{R}$ be a modular function. The total variation $|\mu| : P \to [0, +\infty]$ of μ is defined by $|\mu|(a) := \sup\{\sum_{i=1}^n |\mu(x_i) - \mu(x_{i-1})| : n \in \mathbb{N},\ 0 = x_0 \leq x_1 \leq \ldots \leq x_n = a\}$

We want to stress that without modularity, the total variation of a measure fails to be additive.

Example 3.7. Let L be an orthomodular lattice , $B_1 = \{0, a, b, 1\}$ and $B_2 = \{0, c, d, 1\}$ two blocks such that $L = B_1 \cup B_2$ and $B_1 \cap B_2 = \{0, 1\}$. Let $\mu : L \to \mathbb{R}$ be a measure with $\mu(a) = 3$, $\mu(b) = -1$, $\mu(c) = \mu(d) = 1$. Then $|\mu|(c \vee d) = |\mu|(1) = 4$, but $|\mu|(c) + |\mu|(d) = 2$.

Proposition 3.3. *Let P be a D-lattice and $\mu : P \to \mathbb{R}$ be a measure which is modular.*

(a) *$|\mu|$ is modular.*
(b) *Then $|\mu|(a \oplus b) = |\mu|(a) + |\mu|(b)$ whenever a, b are orthogonal elements of P.*
(c) *$|\mu|(a) = \sup\{\sum_{i=1}^n |\mu(a_i)| : n \in \mathbb{N}, \oplus_{i=1}^n a_i = a\}$ for every $a \in P$.*
(d) *If μ is strongly continuous, then $|\mu|$ is a strongly continuous bounded positive measure.*

Proof:

(a) is contained in (Weber, 1999).
(b) Let $c = a \oplus b$. Then by 1.3.10(a) of (Weber, 1999) we have $|\mu|(c) = |\mu|(a) + d_{|\mu|}(a, c)$, where $d_{|\mu|}(a, c) := \sup\{\sum_{i=1}^n |\mu(x_i) - \mu(x_{i-1})| : n \in \mathbb{N},\ a = x_0 \leq x_1 \leq \ldots \leq x_n = c\}$. We now verify that $d_{|\mu|}(a, c) = |\mu|(b)$. Let $a = x_0 \leq \ldots \leq x_n = c$. Define $y_i := x_i \ominus a$. Then $0 = y_0 \leq \ldots \leq y_n = c \ominus a = b$ and $\sum_{i=1}^n |\mu(x_i) - \mu(x_{i-1})| = \sum_{i=1}^n |\mu(y_i) - \mu(y_{i-1})| \leq |\mu|(b)$. Hence $d_{|\mu|}(a, c) \leq |\mu|(b)$.

    Vice versa, let $0 = y_0 \leq \ldots \leq y_n = b$. Define $x_i := a \oplus y_i$. Then $a = x_0 \leq \ldots \leq x_n = a \oplus b = c$ and $\sum_{i=1}^n |\mu(y_i) - \mu(y_{i-1})| = \sum_{i=1}^n |\mu(x_i) - \mu(x_{i-1})| \leq d_{|\mu|}(a, c)$. Hence $d_{|\mu|}(a, c) \geq |\mu|(b)$.
(c) Let $\mu^*(a) := \sup\{\sum_{i=1}^n |\mu(a_i)| : n \in \mathbb{N}, \oplus_{i=1}^n a_i = a\}$. We will prove $|\mu|(a) \leq \mu^*(a)$ for every $a \in P$.

Let $0 = x_0 \leq x_1 \leq \ldots \leq x_n = a$. Then there exists $d_i \in P$ $(i = 1, \ldots, n)$ such that $x_i = x_{i-1} \oplus d_i$. Then $d_1 \oplus \ldots \oplus d_n = a$ and $\sum_{i=1}^{n} |\mu(x_i) - \mu(x_{i-1})| = \sum_{i=1}^{n} |\mu(d_i)| \leq \mu^*(a)$.

We will prove the reverse inequality.

Let $\oplus_{i=1}^{n} a_i = a$. Put $x_0 := 0$ and $x_i = \oplus_{j=1}^{i} a_j$. Then we have $0 = x_0 \leq x_1 \leq \ldots x_n = a$ and $\sum_{i=1}^{n} |\mu(a_i)| = \sum_{i=1}^{n} |\mu(x_i) - \mu(x_{i-1})| \leq |\mu|(a)$. Therefore $\mu^*(a) \leq |\mu|(a)$.

(d) It follows from the inequality $\|\mu\| \leq |\mu| \leq 2\|\mu\|$ where $\|\mu\|(a) = \sup\{|\mu(x)| : P \ni x \leq a\}$. The former inequality is obvious. We will prove the latter one using (c).

Let $a_i \in P$ such that $\oplus_{i=1}^{n} a_i = a$. We get $\sum_{i=1}^{n} |\mu(a_i)| = \sum_{i=1}^{n} \mu(a_i) \vee 0 + \sum_{i=1}^{n} (-\mu(a_i) \vee 0) = \sum_{i \in I} \mu(a_i) - \sum_{i \in J} \mu(a_i) = \mu(\oplus_{i \in I} a_i) - \mu(\oplus_{i \in J} a_i)$ for some I, J subset of $\{1, 2, \ldots, n\}$. Observe that $\oplus_{i \in I} a_i$ and $\oplus_{i \in J} a_i$ are well-defined elements of P. Moreover, we have $\oplus_{i \in I} a_i \leq a$ and $\oplus_{i \in J} a_i \leq a$. Then $|\mu|(a) \leq 2\|\mu\|(a)$. □

Theorem 3.8. *Let P be a D-lattice with the interpolation property, $\mu : P \to \mathbb{R}^l$ a strongly continuous measure which is modular. Then $\mu(P)$ is convex.*

Proof: Let $\mu = (\mu_1, \ldots, \mu_l)$. We can write $\mu_i = |\mu_i| - (|\mu_i| - \mu_i)$ where $|\mu_i|$ denotes the total variation of μ_i. Write $\mu_i^{**} := |\mu_i|$ and $\mu_i^* := |\mu_i| - \mu_i$. Then μ_i^{**}, μ_i^* are positive measures and by Proposition 3.3 both μ_i^* and μ_i^{**} are strongly continuous. Put $\overline{\mu} := (\mu_1^{**}, \ldots, \mu_n^{**}, \mu_1^*, \ldots, \mu_n^*)$. Then $\overline{\mu}$ is strongly continuous, thus from Theorem 3.6 and Lemma 3.7 $\overline{\mu}(P)$ is convex. Let $T : (z_1, z_2) \in \mathbb{R}^{2l} \mapsto z_1 - z_2 \in \mathbb{R}^l$. Then T is a linear map and $T(\overline{\mu}(P)) = \mu(P)$. It follows that $\mu(P)$ is convex, as a linear image of a convex set. □

We now want to derive from Theorem 3.8 the classical version of Lyapunov theorem where non-atomicity is involved. We start with a definition.

Definition 3.8. Let μ be a function on P. We say that $f \in P$ is a *μ-atom* if $\mu(f) \neq 0$ and for every $g \in P$, $g \leq f$, either $\mu(g) = \mu(f)$, or $\mu(g) = 0$. We say that μ is *atomless* if P does not contain any μ-atoms.

We need the following result.

Proposition 3.4. (Weber, 1996, 2.5) *Let μ be a modular function on a lattice L. Then $N(\mu) := \{(x, y) \in L^2 : \mu$ is constant on $[x \wedge y, x \vee y]\}$ is a congruence relation and the quotient $\hat{L} := L/N(\mu)$ is a modular lattice.*

Proposition 3.5. *Let P a σ-complete D-lattice and $\mu : P \to \mathbb{R}^l$ be a σ-order continuous measure which is modular. Then μ is atomless iff μ is strongly continuous.*

Proof: First, we observe that , by Proposition 2.2, μ is strongly continuous iff P is μ-chained. From 5.8 of (Weber, 1996) P is μ-chained iff $\hat{P} := P/N(\mu)$ is dense-in-itself (i.e. for any $\hat{a}$, $\hat{b} \in \hat{P}$ with $\hat{a} < \hat{b}$ there is an element $\hat{x} \in \hat{P}$ such that $\hat{a} < \hat{x} < \hat{b}$).

We will prove that this latter condition is equivalent to μ atomless. Obviously it is stronger than μ atomless.

Now we prove μ atomless implies $\hat{P}$ dense-in-itself. Let $\hat{a} < \hat{b}$. Replacing a by $a \wedge b$, we may assume $a \leq b$. Let c be such that $b = a \oplus c$. As $\hat{a} \neq \hat{b}$, $\hat{c} \neq \hat{0}$. Since μ is atomless there exists $x \in P$ such that $0 < x < c$, $\mu(x) \neq 0$ and $\mu(x) \neq \mu(c)$. Put $e = x \oplus a$, we get $a < a \oplus x < a \oplus c = b$ and $\mu(a) \neq \mu(e) \neq \mu(b)$. Then $\hat{a} < \hat{e} < \hat{b}$, as claimed. □

From 3.12 and 3.15 it follows

Corollary 3.1. *If P is σ-complete and $\mu : P \to \mathbb{R}^l$ is an atomless σ-order continuous measure which is modular, then its range is convex.*

4. GENERALIZATIONS

In Section 3 we have not used all axioms of a D-poset. It turns out that in 3.3–3.6 we can replace a D-poset with a weaker structure $(L, \leq, \perp, \oplus)$ where $(L, \leq)$ is a poset with 0 and 1, the smallest and the greatest element of L , $\perp$ is a binary relation on L and $\oplus$ is a partially defined binary operation satisfying:

(1) $a \oplus b$ is defined if and only if $a \perp b$;
(2) $a \oplus 0 = 0 \oplus a = a$ for every $a \in L$;
(3) if $a \leq b$ there exists $c \in L$ with $c \perp a$ and $a \oplus c = b$;
(4) if $c' \leq c$, $a' \leq a$ and $c \perp a$, then $c' \perp a'$ and $c' \oplus a' \leq c \oplus a$.

Definition 4.9. We say that μ defined on $(L, \leq, \perp, \oplus)$ is a measure if $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever a, $b \in L$ and $a \perp b$.

Theorem 4.9. *Let $(L, \leq)$ be a poset with the interpolation property, $\perp$ be a binary relation on L and $\oplus$ a partially defined operation on L satisfying the axioms (1)–(4) as above. Let $\mu : L \to \mathbb{R}^l$ be a positive measure such that L is μ-chained.*

(a) Then for all a, $b \in L$ with $a \perp b$ and for all a, $b \in L$ with $a \leq b$ the segment $\overline{\mu(a), \mu(b)}$ is contained in the range of μ; in particular the range of μ is a star-shaped domain with respect to 0.
(b) If L is a lattice and μ is modular, then $\mu(L)$ is convex.

In the proof of Proposition 2.2, Proposition 3.3(b) and therefore in Theorem 3.8, L need satisfy another property, namely

(5) If $a \le x \le a \oplus b$, then there exists $c \le b$ with $a \oplus c = x$.

We will prove the generalization of Proposition 3.3(b).

Proposition 4.6. *Let $(L, \le, \perp, \oplus)$ be a lattice as in Theorem 4.9 satisfying the additional axiom (5). Let $\mu : L \to \mathbb{R}$ be a measure which is modular. Then $|\mu|(a \oplus b) = |\mu|(a) + |\mu|(b)$ whenever $a,\ b$ are orthogonal elements of L.*

Proof: Let $c = a \oplus b$. As in Proposition 3.3(b) we have to verify $d_{|\mu|}(a, c) = |\mu|(b)$.

Let $x_0 := a \le x_1 \le \ldots \le x_{n-1} \le x_n = a \oplus b$. Then since $a \le x_{n-1} \le a \oplus b$, by axiom (5), there exist $y_{n-1} \in L$, $y_{n-1} \le b$ such that $x_{n-1} = a \oplus y_{n-1}$; and so on for every $i = n-2, \ldots, 1$ there exist $y_i \in L$ such that $x_i = a \oplus y_i$ and $y_i \le y_{i+1}$. So for $y_0 := 0 \le y_1 \le y_2 \le \ldots \le y_n := b$, we have $\sum_{i=1}^{n} |\mu(x_i) - \mu(x_{i-1})| = \sum_{i=1}^{n} |\mu(y_i) - \mu(y_{i-1})| \le |\mu|(b)$. Hence $d_{|\mu|}(a, c) \le |\mu|(b)$.

Vice versa goes similarly as in Proposition 3.3(b). □

Theorem 4.10. *Let $(L, \le, \perp, \oplus)$ be a lattice as in Theorem 4.9 satisfying the additional axiom (5). Let $\mu : L \to \mathbb{R}^l$ be a measure which is modular. Suppose that L is a μ-chained lattice with the interpolation property. Then $\mu(L)$ is convex.*

Every complemented modular lattice satisfies properties (1)–(5) of $(L, \le, \perp, \oplus)$ putting $a \oplus b := a \vee b$ whenever $a \wedge b = 0$.

Therefore, Theorem 4.10 also generalizes Avallone's version of Lyapunov's theorem (Theorem 2.3 of Avallone, 1995 for modular functions on complemented lattices. Observe that the convexity of the range of a modular function μ on a complemented lattice L can be reduced to the case that L is modular, passing to the quotient $L/N(\mu)$, (see Proposition 3.4).

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